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On symmetry of solutions of the Yang–Mills equation

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Abstract. Invariant and parameter-invariant solutions of the classical Euclidean SU(2) Yang-Mills equation are studied. It is shown that many interesting solutions of the Yang-Mills equation are parameter invariant and that thereby the solutions with finite action value (instantons) correspond to the parameter-invariant solutions with maximal symmetry. Some new classes of exact solutions are also obtained.

1. Introduction

In the present paper, the solution of the Yang-Mills equation is considered from the group point of view, i.e. its invariant solutions (with respect to some transformation group) are discussed.

As is known (Mack and Salam 1969), the Yang-Mills equation is conformally invariant, i.e. besides the gauge group it admits the conformal group of transformations, whose generators are four translations, six space rotations, dilatation and four special conformal transformations. If, to eliminate the gauge degrees of freedom, we fix the gauge and consider the symmetry of the system of the initial equation with the gauge condition, then the conformal group turns out to be broken up into one of its subgroups. Knowing this symmetry group, the invariant solutions (Ovsyannikov 1962, 1978) of the Yang-Mills equation in the chosen gauge can be found.

However, the possibility of using the invariant solution concept is in a number of cases rather restricted. For example, if we look for the solutions of the Yang-Mills equation

$$\mathscr{D}_\mu G^a_{\mu\nu} = 0 \qquad G^a_{\mu\nu} = A^a_{\nu,\mu} - A^a_{\mu,\nu} + g\varepsilon_{abc}A^b_\mu A^c_\nu$$

which are invariant under the group of translations with generators $X_\alpha = \partial/\partial x_\alpha$ $(\alpha = 1, \ldots, 4)$ in the form

$$\psi(x, A) = 0$$

then the condition of invariance

$$X_\alpha \psi(x, A) = 0$$

means clearly that the sought solution has only a trivial form: $A^a_\mu = \text{constant}$. In terms of the theory of the group properties of differential equations (Ovsyannikov 1962, 1978) such a situation is quite natural. If

$$X_\alpha = \xi^i_\alpha(x, A)\frac{\partial}{\partial x_i} + \eta^k_\alpha(x, A)\frac{\partial}{\partial A^k}$$

$$(\alpha = 1, \ldots, r \qquad i = 1, \ldots, n \qquad k = 1, \ldots, m)$$

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are the basis operators of the Lie algebra of the r-parameter group H (n and m are numbers of variables and functions respectively), then the equations for the invariant H solution are

$$\Phi^k(I_1, \ldots, I_t) = 0 \qquad k = 1, \ldots, m \qquad t = N - R = n + m - R \qquad (1)$$

where

$$R = \operatorname{rank} \|\xi^i_\alpha, \eta^k_\alpha\|$$

and I_τ are invariants;

$$X_\alpha I_\tau = 0 \qquad \tau = 1, \ldots, t.$$

Thus, if $R = n$, then $t = m$ and equations (1) take the form

$$I_\tau = C_\tau = \text{constant}.$$

(In the simple example considered $R = n = 4$ and $A^a_\mu = \text{constant}$.)

Therefore it might be worthwhile to somewhat modify the invariant solution concept and to introduce parameter-invariant solutions which depend not only on the initial variables (x_μ) but also on the parameters (a_α). Thereby to each group generator X_α corresponds its parameter a_α.

The invariant solution does not change its form under the transformation of the group. The parameter-invariant solution transforms to a new solution which differs from the initial one by the value of the parameter and reduces to the initial solution by the parameter redefinition. Thus the parameter-invariant solution may be regarded as an invariant one, considering herewith that on the transformation X_α of the group H the corresponding parameter changes as well. For an Abelian group H (for instance, for each one-parameter group) the transformation rule of a_α may be chosen in the form $\partial/\partial a_\alpha$ (in this case, a_α is the canonical parameter). We shall choose the transformation rule of a_α on the basis of that of x_α (for example, if a_α is the vector, then $\xi_a = \xi_x$ $(x \to a)$).

In this manner, by introducing a set of parameters the number of independent variables of the manifold can be increased from n to $n + r$, thereby the number of independent invariants I_τ grows as well: $t \to t + r = n + m - R + r$ and in the case of $R = n$, $t = m + r > m$, system (1) will possess non-trivial solutions.

The process of increasing the number of variables may be continued by assigning several parameters to each generator of the symmetry group, which leads to the further growth of the number of independent invariants in expression (1).

The present paper is devoted to the discussion of the parameter-invariant solutions of the classical Yang-Mills equation in the Lorentz gauge. In § 2, the invariant and parameter-invariant solutions of the Yang-Mills equation are found for the case when each generator of the symmetry group corresponds to not more than one parameter. Section 3 deals with some parameter-invariant solutions in the case when, to each generator of the group correspond a number of parameters (or, more exactly, when each considered subgroup of the symmetry group corresponds to several parameters with some transformational properties (scalars, vectors, tensors)).

2. One-parameter-invariant solutions

Let us study the solutions of the classical Euclidean SU(2) Yang-Mills equation in the Lorentz gauge

$$\partial_\mu G^a_{\mu\nu} + g\varepsilon_{abc} A^b_\mu G^c_{\mu\nu} = 0$$
$$\partial_\mu A^a_\mu = 0 \qquad (2)$$
$$\mu, \nu = 1, \ldots, 4 \qquad a, b, c = 1, 2, 3.$$

System (2) admits the group H of point transformations (Lie point symmetry group), the infinitesimal operators of which (Rosenhaus and Kiiranen 1982) are

$$X_\mu = \frac{\partial}{\partial x_\mu}$$
$$X_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} + A^a_\mu \frac{\partial}{\partial A^a_\nu} - A^a_\nu \frac{\partial}{\partial A^a_\mu} \qquad (3)$$
$$X_{ab} = A^a_\mu \frac{\partial}{\partial A^b_\mu} - A^b_\mu \frac{\partial}{\partial A^a_\mu}$$
$$X = x_\mu \frac{\partial}{\partial x_\mu} - A^a_\mu \frac{\partial}{\partial A^a_\mu}.$$

Let us look for the parameter-invariant solutions of system (2) (with respect to certain subgroups of the group H) in the explicit form

$$A^a_\mu - f^a_\mu(x, b_i) = 0 \qquad (4)$$

where b_i are parameters.

For the translational subgroup $\{X_\mu\}$ with generators

$$X_\mu = \frac{\partial}{\partial x_\mu}$$

the $\{X_\mu\}$-invariant solutions (as noted before) are only

$$A^a_\mu = c^a_\mu = \text{constant} \qquad c^a_\mu c^b_\nu = c^a_\nu c^b_\mu$$

corresponding to $G^a_{\mu\nu} = 0$ (and to the zero action value: $S = 0$).

Excluding from consideration these trivial vacuum solutions, let us proceed to the $\{X_\mu\}$-parameter-invariant solutions. Introducing the translational parameters c_μ, we construct new generators

$$X_\mu \to X'_\mu = \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial c_\mu}$$

(in the given case, c_μ is the canonical parameter). The form of the parameter-invariant solution is determined from

$$X'_\alpha(A^a_\mu - f^a_\mu(x, b)) = 0$$

and $\{X'_\mu\}$-invariant ($\{X_\mu\}$-parameter-invariant) solution takes the well known form

$$A^a_\mu - f^a_\mu(x - c, b) = 0. \qquad (5)$$

Now we shall consider in succession the invariance (the parametric invariance) under the other subgroups of the group H with generators (3). Respectively, in all further expressions x_α means $x_\alpha - c_\alpha$.

2.1. The subgroup of rotations in the coordinate and functional spaces with generators $X_{\mu\nu}$

It can be shown easily that the invariance condition

$$X_{\alpha\beta}(A^a_\mu - f^a_\mu)|_{A^a_\mu = f^a_\mu} = 0$$

for system (2) cannot be satisfied, i.e. system (2) has no $\{X_{\mu\nu}\}$-invariant solutions.

Let us find parameter-invariant solutions.

(i) Let the parameter corresponding to the subgroup $\{X_{\alpha\beta}\}$ be the vector (with respect to the coordinate space) R^a_α, i.e.

$$X_{\alpha\beta} \to X'_{\alpha\beta} = x_\alpha \frac{\partial}{\partial x_\beta} - x_\beta \frac{\partial}{\partial x_\alpha} + A^a_\alpha \frac{\partial}{\partial A^a_\beta} - A^a_\beta \frac{\partial}{\partial A^a_\alpha} + R^a_\alpha \frac{\partial}{\partial R^a_\beta} - R^a_\beta \frac{\partial}{\partial R^a_\alpha}.$$

The condition of parametric invariance

$$X'_{\alpha\beta}(A^a_\mu - f^a_\mu) = 0 \tag{6}$$

defines the possible form of the solution

$$A^a_\mu = R^a_\alpha x_\alpha x_\mu g(x^2) \tag{7}$$

or

$$A^a_\mu = R^a_\mu f(x^2). \tag{7'}$$

However, expression (7) determines the Abelian field: $[A_\mu, A_\nu] = 0$, and expression (7′) contradicts the gauge condition of system (2).

Thus, for the case (i) there is no parameter-invariant solution in the Lorentz gauge.

(ii) Consider now the parameter-invariant solution with the tensor parameter $T^a_{\alpha\beta}$:

$$X_{\alpha\beta} \to X'_{\alpha\beta} = X_{\alpha\beta} + T^a_{\alpha\gamma} \frac{\partial}{\partial T^a_{\beta\gamma}} - T^a_{\beta\gamma} \frac{\partial}{\partial T^a_{\alpha\gamma}}.$$

Now from the condition of parametric invariance (6) it follows that

$$A^a_\mu = T^a_{\mu\nu} x_\nu f(x^2). \tag{8}$$

Substituting expression (8) into the second equation of system (2), we obtain

$$T^a_{\mu\mu} f + 2x_\mu x_\nu T^a_{\mu\nu} f' = 0. \tag{9}$$

Let us expand $T^a_{\mu\nu}$ on the symmetric and antisymmetric components (with respect to μ and ν)

$$T^a_{\mu\nu} = S^a_{\mu\nu} + A^a_{\mu\nu}.$$

If $S^a_{\mu\nu} \neq 0$, then equation (9) gives

$$S^a_{\mu\nu} \sim \delta_{\mu\nu} \qquad f = c/(x^2)^2.$$

Excluding from consideration the Abelian (vacuum) configuration we have

$$T^a_{\mu\nu} = -T^a_{\nu\mu}.$$

The substitution of (8) into the Yang–Mills equation (2) results in the specification of the form of the tensor $T^a_{\mu\nu}$. As is well known,

$$T^a_{\mu\nu} = \eta^a_{\mu\nu} \qquad (T^a_{\mu\nu} = \bar{\eta}^a_{\mu\nu})$$

where $\eta^a_{\mu\nu}(\bar{\eta}^a_{\mu\nu})$ are 't Hooft symbols ('t Hooft 1976)

$$(\eta^a_{mn} = \bar{\eta}^a_{mn} = \varepsilon_{amn},\ \eta^a_{\mu 4} = -\bar{\eta}^a_{\mu 4} = \delta_{a\mu},\ \eta^a_{\nu\mu} = -\eta^a_{\mu\nu},\ \bar{\eta}^a_{\nu\mu} = -\bar{\eta}^a_{\mu\nu}).$$

Thus, the parameter-invariant $\{X_{\mu\nu}\}$ solutions with the tensor parameter $T^a_{\mu\nu}$ have the form

$$A^a_\mu = \eta^a_{\mu\nu} x_\nu f(x^2). \tag{10}$$

In a general case, parameter-invariant solutions with tensor parameters of an arbitrary rank are also possible.

(iii) $A^a_\mu = T^a_{\mu\alpha\beta} x_\alpha x_\beta f(x^2)$.

Representing $T^a_{\mu\alpha\beta}$ as a sum of symmetric and antisymmetric (with respect to the last two indices) parts and taking account of the gauge condition (2) brings the given solution to (7′).

Analogously, the solution of the form

$$A^a_\mu = T^a_{\mu\alpha\beta\gamma} x_\alpha x_\beta x_\gamma f(x^2)$$

leads to (8). The absence of new parameter-invariant solutions in more complicated cases is also obvious.

2.2. The subgroup of rotations in the isospace $\{X_{ab}\}$

The solutions of form (10) are, evidently, $\{X_{ab}\}$ parameter invariant if the corresponding parameter is the same, i.e. $\eta^a_{\mu\nu}$,

$$X_{ab} \to X'_{ab} = A^a_\mu \frac{\partial}{\partial A^b_\mu} - A^b_\mu \frac{\partial}{\partial A^a_\mu} + \eta^a_{\mu\nu} \frac{\partial}{\partial \eta^b_{\mu\nu}} - \eta^b_{\mu\nu} \frac{\partial}{\partial \eta^a_{\mu\nu}}.$$

2.3. The dilatation subgroup $\{X\}$

$$X = x_\mu \frac{\partial}{\partial x_\mu} - A^a_\mu \frac{\partial}{\partial A^a_\mu}.$$

Let us first find the $\{X\}$-invariant solutions in the form (10):

$$X(A^a_\mu - \eta^a_{\mu\nu} x_\nu f(x^2)) = 0$$

from which the form of the dilatation-invariant solution

$$A^a_\mu = k\eta^a_{\mu\nu} x_\nu / x^2 \qquad k = \text{constant}. \tag{11}$$

Inserting (11) into the Yang–Mills equation (2), we obtain

$$k = 1/g.$$

The corresponding solution is the meron (De Alfaro *et al* 1976) (meron–meron pair with singularities at $|x| = 0$ and $|x| = \infty$).

Consider now the $\{X\}$-parameter-invariant solutions:

$$X \to X' = x_\mu \frac{\partial}{\partial x_\mu} - A^a_\mu \frac{\partial}{\partial A^a_\mu} + \rho \frac{\partial}{\partial \rho}$$

(where ρ is the parameter).

Using the invariance condition, we get

$$A^a_\mu = \frac{2}{g} \eta^a_{\mu\nu} \frac{x_\nu}{x^2} \varphi\left(\frac{x^2}{\rho^2}\right) \tag{12}$$

(the coefficient $2/g$ is separated for convenience). Substituting (12) into (2), we obtain

$$\dot\varphi^2 = \varphi^2(1-\varphi)^2 + c_1$$
$$\dot\varphi = \frac{\partial\varphi}{\partial t} = z\frac{\partial\varphi}{\partial z} \qquad t = \ln z \qquad z = x^2/\rho^2 \qquad c_1 = \text{constant} \tag{13}$$

whereas the action takes the form

$$S = \tfrac{1}{4}\int d^4x\, G^a_{\mu\nu}\,\tilde G^a_{\mu\nu} = (24\pi^2/g^2)\int dt[\dot\varphi^2 + \varphi^2(1-\varphi)^2].$$

In the simplest case, $c_1 = 0$, we obtain two solutions of equation (13):

$$\varphi = z/(1+z) \qquad \varphi = 1/(1+z)$$

and two Yang-Mills configurations, respectively:

$$A^a_\mu = \frac{2}{g}\eta^a_{\mu\nu}\frac{x_\nu}{x^2+\rho^2} \qquad A^a_\mu = \frac{2}{g}\eta^a_{\mu\nu}\frac{x_\nu\rho^2}{x^2(x^2+\rho^2)}. \tag{14}$$

Expressions (14) describe the instanton (Belavin *et al* 1975) (in the regular gauge) and the anti-instanton (in the singular gauge), respectively.

By introducing $\varphi = u + \frac{1}{2}$ the solutions for a more general case are also easily obtained:

$$-\tfrac{1}{16} \leqslant c_1 \leqslant 0 \qquad 0 \leqslant k \leqslant 1$$
$$u = [k^2/2(1+k^2)]^{1/2}\,\mathrm{sn}[t/[2(1+k^2)]^{1/2}, k] \tag{15}$$

where $\mathrm{sn}[u, k]$ is the Jacobi elliptic sine of the parameter k. Formulae (15) have been obtained by Baseyan and Matinyan (1980).

There also exist the singular elliptic solutions of (13) (Rosenhaus 1986).

$$c_1 \leqslant -\tfrac{1}{16} \qquad 0 \leqslant k < \sqrt{2}/2$$
$$u = [(1-k^2)/2(1-2k^2)]^{1/2}/\mathrm{cn}[t/[2(1-2k^2)]^{1/2}, k] \tag{16a}$$
$$c_1 = -\tfrac{1}{16}\,(k=0) \qquad u = (\sqrt{2}\cos(t/\sqrt{2}))^{-1}.$$

$$-\tfrac{1}{16} < c_1 \leqslant 0 \qquad 0 < k \leqslant 1$$
$$u = ([2(1+k^2)]^{1/2}\,\mathrm{sn}[t/[2(1+k^2)]^{1/2}, k])^{-1}. \tag{16b}$$

$$c_1 \geqslant 0 \qquad \sqrt{2}/2 < k \leqslant 1$$
$$u = (1 - \mathrm{cn}[t/(2k^2-1)^{1/2}, k])/((2k^2-1)^{1/2}\,\mathrm{sn}[t/(2k^2-1)^{1/2}, k]). \tag{16c}$$

Note that for each solution (15) and (16) there also exists the corresponding solution with $u \to -u(\varphi \to 1-\varphi)$ which has the same action value S and the opposite sign of topological charge

$$Q = \frac{g^2}{32\pi^2}\int G^a_{\mu\nu}\,\tilde G^a_{\mu\nu}\,dx^4 = 6\int dt\,\dot\varphi\varphi(1-\varphi).$$

3. Multiparameter-invariant solutions

Consider now the case when more than one parameter corresponds to different subgroups of the symmetry group. Let the number of translational parameters be equal to two: a, b

$$X'_\mu = \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial a_\mu} + \frac{\partial}{\partial b_\mu}.$$

Thus, now we have two parametric invariants

$$y = x - a \qquad z = x - b. \tag{17}$$

As above, we shall consider the invariance (the parametric invariance) in succession under different subgroups of symmetry group (3), whereby we again assume that to the generators of the rotation group $X_{\mu\nu}$ corresponds the only tensor parameter $T^a_{\mu\nu}(\eta^a_{\mu\nu})$, so that the sought solution will be

$$A^a_\mu = \eta^a_{\mu\nu}\{y_\nu \varphi(y^2, yz, z^2) + z_\nu \psi(y^2, yz, z^2)\} \qquad (yz \equiv y_\alpha z_\alpha). \tag{18}$$

3.1.

First, we demand the invariance of (18) under the dilatation subgroup with the operator

$$X = x_\alpha \frac{\partial}{\partial x_\alpha} - A^a_\alpha \frac{\partial}{\partial A^a_\alpha} \to y_\alpha \frac{\partial}{\partial y_\alpha} + z_\alpha \frac{\partial}{\partial z_\alpha} - A^a_\alpha \frac{\partial}{\partial A^a_\alpha}. \tag{19}$$

The invariance condition is fulfilled if

$$\begin{aligned} \varphi &= \frac{1}{y^2}\bar{\varphi}(y^2/yz, z^2/yz) \\ \psi &= \frac{1}{z^2}\bar{\psi}(y^2/yz, z^2/yz). \end{aligned} \tag{20}$$

We shall, however, look for the solution of the Yang-Mills equation in a less general form (Corrigan and Fairlie 1977)

$$A^a_\mu = -\frac{1}{g}\eta^a_{\mu\nu}\partial_\nu \ln \Phi(y^2, yz, z^2) \tag{21}$$

where only one vector $\partial/\partial x_\nu$ is introduced instead of two: y_ν and z_ν. The dilatational invariance of (21) means that the function Φ must be the homogeneous function of its arguments, i.e.

$$\Phi = (yz)^k f\left(\frac{y^2}{yz}, \frac{z^2}{yz}\right) \qquad k = \text{constant}. \tag{22}$$

It is known (Corrigan and Fairlie 1977) that (21) is the solution of the Yang-Mills equation if Φ satisfies the equation

$$\partial_\mu\partial_\mu\Phi = -c\Phi^3 \qquad c = \text{constant}. \tag{23}$$

Substituting (22) into (23), it may be concluded that the only case to discuss is $k = -1$ and for f we have

$$\begin{aligned} &2f + 4(\alpha f_\alpha + \beta f_\beta) + [\alpha^2 f_{\alpha\alpha} + 2(\alpha\beta - 2)f_{\alpha\beta} + \beta^2 f_{\beta\beta}] = -\bar{c}f^3 \\ &\alpha = y^2/yz \qquad \beta = z^2/yz \qquad \bar{c} = c/(a-b)^2. \end{aligned} \tag{24}$$

Let us restrict ourselves to the solutions of equation (24) only in the simplest case

$$\begin{aligned} &2f + 4(\alpha f_\alpha + \beta f_\beta) + (\alpha^2 f_{\alpha\alpha} + 2\alpha\beta f_{\alpha\beta} + \beta^2 f_{\beta\beta}) = 0 \\ &f_{\alpha\beta} = \tfrac{1}{4}\bar{c}f^3. \end{aligned} \tag{25}$$

If $B \equiv \alpha f_\alpha + \beta f_\beta + f$, then the first equation of system (25) is

$$\alpha B_\alpha + \beta B_\beta = -2B$$

and

$$B = \alpha^{-2}\psi(\alpha/\beta).$$

Finding f from here and inserting it into the second equation of (25), we get

$$f = (1/\alpha)\varphi(\alpha/\beta) - \alpha^{-2}\psi(\alpha/\beta). \tag{26}$$

(i) $c = 0$.

$$\varphi = \frac{c_1}{y^2} + \frac{k_1}{z^2} + c_2\frac{yz}{y^4} + k_2\frac{yz}{z^4} \qquad (c_i,\, k_i = \text{constant}).$$

Choosing $c_2 = k_2 = 0$, $c_1 = \lambda_1^2$, $k_1 = \lambda_2^2$ we obtain the one-instanton solution in the form by Jackiw *et al* (1977) although such a choice actually means the introducing of the dilatational parameters and will be discussed below.

(ii) $c \neq 0$. In this case, $\psi = 0$ and φ satisfies

$$\varphi'' + \frac{\varphi^3}{\bar{x}^3} = 0$$
$$\varphi = \varphi(\bar{x}) \qquad \bar{x} = \frac{\alpha}{\beta} = \frac{y^2}{z^2} = \frac{(x-a)^2}{(x-b)^2}. \tag{27}$$

The corresponding Φ takes the form

$$\Phi = \frac{\bar{a}}{y^2}\varphi\left(\frac{y^2}{z^2}\right) \qquad \bar{a} = \text{constant} \tag{28}$$

which coincides with the form of the ansatz by Cervero *et al* (1977). The obvious solution of (27):

$$\varphi = \tfrac{1}{2}\sqrt{\bar{x}}$$

correspond to

$$\Phi = \tfrac{1}{2}\bar{a}(y^2z^2)^{-1/2}. \tag{29}$$

Equation (29) is the meron–antimeron pair with singularities in two arbitrary points $x = a$ and $x = b$ (De Alfaro *et al* 1976).

To solve (27) like Cervero *et al* (1977) we introduce

$$\varphi = 2\bar{x}^{1/2}v(\bar{y}) \qquad \bar{y} = \tfrac{1}{2}\ln\bar{x} = \tfrac{1}{2}\ln(y^2/z^2). \tag{30}$$

Then

$$\Phi \sim (y^2z^2)^{-1/2}v[\tfrac{1}{2}\ln(y^2/z^2)] \tag{30$'$}$$

where v satisfies the equation

$$v'' - v + v^3 = 0 \qquad \sqrt{2}v' = (-v^4 + 2v^2 + \bar{c})^{1/2} \qquad \bar{c} \geqslant -1 \tag{31}$$

($\bar{c} = -1$ corresponds to solution (29)). The solutions of (31) are again expressed through the Jacobi elliptic functions.

$$\bar{c} \geqslant 0 \qquad \sqrt{2}/2 < k \leqslant 1$$
$$v = k(k^2 - \tfrac{1}{2})^{-1/2}\text{cn}[(\bar{y} + c_1)/2(2k^2-1)^{1/2},\, k] \qquad (c_1 = \text{constant}). \tag{32}$$

$$-1 \leqslant \bar{c} \leqslant 0 \qquad 0 \leqslant k \leqslant 1$$

$$v = \left(\frac{2}{2-k^2}\right)^{1/2} \mathrm{dn}[(\bar{y}+c_1)/(2-k^2)^{1/2}, k]. \tag{33}$$

Note that expression (33) is consistent with that by Cervero *et al* (1977) only if by k (k') in expression (8) of the cited paper one understands k^2 ($k'^2 = 1-k^2$), where k is the parameter of the elliptic functions.

3.2.

Let us move on to parameter-invariant solutions under a dilatation subgroup. Let ρ be the only parameter of this group,

$$X \to X' = y_\alpha \frac{\partial}{\partial y_\alpha} + z_\alpha \frac{\partial}{\partial z_\alpha} - A^a_\alpha \frac{\partial}{\partial A^a_\alpha} + \rho \frac{\partial}{\partial \rho}.$$

Restricting ourselves again to the form of solution (21), instead of (18) we have

$$\Phi = (\rho^2)^k f(y^2/\rho^2, yz/\rho^2, z^2/\rho^2) \qquad k = \text{constant}. \tag{34}$$

Note the essential difference from the preceding case: the arguments of the function f are not independent any more:

$$\frac{(y-z)^2}{\rho^2} = \frac{l^2}{\rho^2} = S$$

where S is a scalar parameter independent of dilatation. Therefore

$$\frac{2yz}{\rho^2} = \alpha + \beta - S \tag{35}$$

and

$$\Phi = (\rho^2)^k f(\alpha, \beta) \qquad \alpha = y^2/\rho^2 \qquad \beta = z^2/\rho^2. \tag{34'}$$

Inserting (34′) into (23) we obtain

$$\alpha f_{\alpha\alpha} + \beta f_{\beta\beta} + (\alpha+\beta-S) f_{\alpha\beta} + 2(f_\alpha + f_\beta) = -\tfrac{1}{4} c (\rho^2)^{3k+1} f^3. \tag{36}$$

As before, we restrict ourselves only to

$$\begin{aligned} &(\alpha f_\alpha + \beta f_\beta + f)_{,\alpha} + (\alpha f_\alpha + \beta f_\beta + f)_{,\beta} = 0 \\ &f_{\alpha\beta} = \bar{S} f^3 \qquad (\bar{S} = \tfrac{1}{4} c (\rho^2)^{3k+1}/S = \text{constant}). \end{aligned} \tag{37}$$

We find:

$$f = \varphi(\alpha-\beta) + (1/\alpha)\psi(\alpha/\beta) \qquad \varphi'' + \psi''/\beta^3 = \bar{S}(\varphi + \psi/\alpha)^3. \tag{38}$$

If $c = 0$ ($\bar{S} = 0$), then

$$f = c_1 + c_2 \frac{y^2 - z^2}{\rho^2} + k_1 \frac{\rho^2}{y^2} + k_2 \frac{\rho^2}{z^2}.$$

The choice $c_1 = c_2 = 0$ gives an anti-instanton solution in the form by Jackiw *et al* (1977) and the choice $c_2 = 0$, $c_1 = k_1 = k_2 = 1$ leads to

$$f = 1 + \frac{\rho^2}{y^2} + \frac{\rho^2}{z^2}. \tag{39}$$

Equation (39) is the two-anti-instanton solution in the singular gauge (in the form by 't Hooft-Witten (Witten 1977), with the same sizes of instantons).

The case $c \neq 0$ leads again to solution (33).

In a more general case of the dependence on two dilatational parameters and also in the general case of the dependence on n translation and m dilatational parameters ($m \leq n$), the generalisations are evident:

$$f = 1 + \sum_{i=1}^{n} \frac{\rho_i^2}{y_i^2} \qquad y_i \equiv x - a_i.$$

It is known that for each n-anti-instanton solution of form (21) there exists the corresponding n-instanton solution, when changing $\eta^a_{\mu\nu} \to \bar{\eta}^a_{\mu\nu}$.

Let us now discuss some invariant solutions in a form different from (8). We assume that the two parameters correspond to the rotation subgroup $\{X_{\alpha\beta}\}$ (in a general case, the number of parameters may be larger than two). Let these parameters be η^a_μ, k_μ:

$$X'_{\alpha\beta} = X_{\alpha\beta} + \eta^a_\alpha \frac{\partial}{\partial \eta^a_\beta} - \eta^a_\beta \frac{\partial}{\partial \eta^a_\alpha} + k_\alpha \frac{\partial}{\partial k_\beta} - k_\beta \frac{\partial}{\partial k_\alpha}.$$

Then the form of the $\{X'_{\alpha\beta}\}$-invariant solution is

$$A^a_\mu = \frac{1}{g} \eta^a_\mu f(kx, x^2) \tag{40}$$

or

$$A^a_\mu = \frac{1}{g} \eta^a_\alpha x_\alpha x_\mu f(kx, x^2) \tag{41}$$

whereby, however, solution (41) defines an Abelian field.

Substituting (40) into the second equation of (2) we have

$$f = f(kx/\sqrt{k^2}) \qquad \eta^a_\mu k_\mu = 0 \tag{40'}$$

and into the Yang-Mills equation (2),

$$\eta^a_\mu \eta^a_\nu = l\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$
$$f'^2 = c_1 + lf^4 \qquad f' = \mathrm{d}f/\mathrm{d}\bar{x} \qquad \bar{x} = kx/\sqrt{k^2} \qquad c_1 = \text{constant}. \tag{40''}$$

Choosing $l = 1$, we obtain

$c_1 = 0$

$$f = \pm 1/\bar{x} \tag{42a}$$

$c_1 < 0 \qquad a = (|c_1|)^{1/4}$

$$f = \pm \frac{a}{\mathrm{cn}(a\sqrt{2}\bar{x}, 1/\sqrt{2})} \tag{42b}$$

$c_1 > 0$

$$f = \pm a\left(\frac{1 - \mathrm{cn}(\sqrt{2}a\bar{x}, 1/\sqrt{2})}{1 + \mathrm{cn}(\sqrt{2}a\bar{x}, 1/\sqrt{2})}\right). \tag{42c}$$

As far as the vector η^a_μ is concerned, equations (40′) and (40″) are satisfied by

$$\eta^a_\mu = \eta^a_{\mu\alpha} \frac{k_\alpha}{\sqrt{k^2}}. \tag{43}$$

Thus the $\{X_{\alpha\beta}\}$-parameter-invariant solution (with parameters η^α_μ, k_μ) has the form

$$A^a_\mu = \frac{1}{g} \eta^a_{\mu\alpha} \frac{k_\alpha}{\sqrt{k^2}} f\left(\frac{kx}{\sqrt{k^2}}\right) \tag{44}$$

with the function f determined by expressions (42).

4. Concluding remarks

In the present paper, in finding the exact solutions of the Yang–Mills equation we have required invariance (parametric invariance) under all generators of the symmetry group and the solutions, invariant under some subgroups of this group, have not been considered. We have discussed only the solutions with one certain type of symmetry, when to each generator there corresponds one or more parameters with the same transformational properties. For instance, we have not studied the solutions of the form

$$A^a_\mu = R^a_\mu f_1(x^2) + T^a_{\mu\nu} x_\nu f_2(x^2)$$

or other solutions with 'mixed' symmetry (see, for example, Irshadullah Khan 1984). We have also restricted ourselves only to the simplest cases of parameter-invariant solutions.

Nevertheless, as was shown above, in the case of a rather complicated physical system, the symmetry properties also simplify the finding of the solution required and many interesting solutions turn out to be parameter invariant. In that sense the finite action solutions (instantons) correspond to parameter-invariant solutions with maximal symmetry.

It should be noted, however, that multi-instanton solutions of the general form (Atiyah *et al* 1978) have not been considered in the present paper.

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